

Quadratic superconducting cosmic strings revisited^(a)

MUSTAPHA AZREG-AÏNOU^(b)

Başkent University, Engineering Faculty, Bağlıca Campus, Ankara, Turkey

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Abstract. – It has been shown that 5-dimensional general relativity action extended by appropriate quadratic terms admits a singular superconducting cosmic string solution. We search for cosmic strings endowed with similar and extended physical properties by directly integrating the non-linear matrix field equations thus avoiding the perturbative approach by which we constructed the above-mentioned *exact* solution. The most general superconducting cosmic string, subject to some constraints, will be derived and shown to be mathematically *unique* up to linear coordinate transformations mixing its Killing vectors. The most general solution, however, is not globally equivalent to the old one due to the existence of Killing vectors with closed orbits.

Introduction. – In d -dimensional general relativity (GR) or Kaluza-Klein (KK) theories ($d > 4$), one can generate non-linear electrodynamics just by coupling the usual scalar curvature to quadratic and higher order terms in the curvature making up the Gauss-Bonnet (GB) term. It has been shown that for ($d > 4$) the inclusion of these extra terms in the Lagrangian leads to modified, most general field equations including up to second-order derivatives of the metric [1], which upon $4+(d-4)$ dimensional reduction split into a set of equations governing non-linear 4-dimensional GR coupled to both non-linear electrodynamics and scalar fields [2–8]. The effective reduced equations are not equivalent to those obtained upon coupling 4-dimensional GR to non-linear electrodynamics [9–11], nor are they equivalent to those derived from higher dimensional Einstein-Maxwell or Einstein-Born-Infeld theories modified by GB term [12–15].

Some solutions to these extended à la GB KK theories under the spherical and cylindrical symmetry assumptions were constructed and interpreted as singular static black holes [7] and singular 4-stationary neutral, charged or superconducting cosmic strings as well as multiple cosmic strings [8], respectively. Regular wormhole-type solutions have been constructed too [14, 16, 17], some of which do not violate the weak energy condition [16] and others do not violate any energy condition [17].

Generalized à la GB GR theories have gained their sharp appeal these last three years with ever-decreasing efforts in dealing with up-to-date problems of black holes, brane world models, gravitational dark energy, cosmic acceleration and string theory [14, 15, 17–25]. This parallels a renewed interest in string theory fueled by the discovery of Capodimonte-Sternberg-Lens Candidate, CSL-1, a pair of aligned galaxies which lie at a redshift of $z = 0.46$ whose double image could be a result of gravitational lensing caused by a cosmic string [26]. The connection is that the GB Lagrangian (GBL) appears in the low energy limit of string theory and its linearized form (GBL) leads upon quantization to ghost-free theory [2, 22].

In the second section the KK theory extended by GB term is introduced and the field equations under the cylindrical symmetry assumption are derived. The cosmic-string solutions obtained so far are discussed and classified. In the third section we derive by a cognitive approach the most general superconducting cosmic-string solution subject to some constraints and prove its unicity up to similarity transformations. In the fourth section we discuss the physical content of the new solution and conclude in the last section.

Quadratic Kaluza-Klein theory. – In 5 - dimensional GR the most general field equations containing at most second order partial derivatives of the metric write upon ignoring a cosmological term

$$R_{AB} - (1/2)Rg_{AB} + \gamma L_{AB} = 0, \quad (1)$$

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^(b)azreg@baskent.edu.tr

where γ is a constant and L_{AB} is the covariantly conserved ($L^{AB}{}_{;A} = 0$) Lanczos tensor associated with the GBL density \mathcal{L}

$$L_{AB} \equiv R_A{}^{CDE}R_{BCDE} - 2R^{CD}R_{ACBD} - 2R_{AC}R_B{}^C + RR_{AB} - (1/4)g_{AB}\mathcal{L}, \quad (2)$$

with $\mathcal{L} \equiv R^{ABCD}R_{ABCD} - 4R^{AB}R_{AB} + R^2$ (R_{ABCD} & R_{AB} are the 5-dimensional Riemann & Ricci tensors and $A, B, C, D, E = 1, \dots, 5$).

From now on we restrict ourselves to stationary cylindrically symmetric 5-dimensional metrics. Such metrics have four commuting Killing vectors $\xi_a{}^A = \delta_a{}^A$ ($a, b = 2, \dots, 5$), one of which ($\xi_4{}^A$) is timelike, and two of which ($\xi_2{}^A$ & $\xi_5{}^A$) have closed orbits. Using the coordinates adapted to these vectors, the 5-metric can be parameterized as

$$ds^2 = -d\rho^2 + \lambda_{ab}(\rho) dx^a dx^b, \quad (3)$$

where $x^2 = \varphi$ & x^5 are periodic, $x^4 = t$ is timelike, $x^3 = z$ and ρ is a radial coordinate. $\lambda_{ab}(\rho)$ is a 4×4 real symmetrical matrix of signature $(- - + -)$ so that the 5-metric has a Lorentzian signature $(- - - +)$. Introducing the matrices $\chi \equiv \lambda^{-1}\lambda_{,\rho}$ & $B \equiv \chi_{,\rho} + (1/2)\chi^2$, eqs. (1) reduce to a system of non-linear scalar & matrix differential equations

$$6 \text{Tr} B + (\text{Tr} \chi)^2 - \text{Tr} \chi^2 + \gamma \{ \text{Tr}(B\chi^2) - \text{Tr}(B\chi)\text{Tr} \chi + (1/2)\text{Tr} B[(\text{Tr} \chi)^2 - \text{Tr} \chi^2] \} = 0, \quad (4)$$

$$2\chi_{,\rho} + 4\text{Tr} \chi_{,\rho} + (\text{Tr} \chi)\chi + \text{Tr} \chi^2 + (\text{Tr} \chi)^2 + \gamma \{ (\chi^3)_{,\rho} - (\text{Tr} \chi)(\chi^2)_{,\rho} + [(\text{Tr} \chi)^2 - \text{Tr} \chi^2]\chi_{,\rho} - (\text{Tr} \chi_{,\rho})[\chi^2 - (\text{Tr} \chi)\chi] - (1/2)[(\text{Tr} \chi^2)_{,\rho}\chi - (\text{Tr} \chi)^3\chi] + (1/2)[(\text{Tr} \chi)\chi^3 - (\text{Tr} \chi^2)\chi^2 - (\text{Tr} \chi)(\text{Tr} \chi^2)\chi] \} = 0. \quad (5)$$

The system (4 & 5) remains invariant if one performs a linear coordinate transformation with constant coefficients mixing the four commuting Killing vectors together and their associated cyclic coordinates

$$x^a = S^a{}_b x_N^b, \quad (6)$$

where $S^a{}_b$ is a constant real matrix. Here x^a & x_N^b are the old and new coordinates, respectively. Such a transformation is equivalent to a similarity transformation on χ ($\chi = S\chi_N S^{-1}$). Solutions related by such transformations belong actually to the same class of equivalence. However, when some Killing vectors have closed orbits, say $\xi_2{}^A$ and $\xi_5{}^A$ in our case, it is possible to generate new solutions which are not globally equivalent to old ones if one at least of these two vectors is re-scaled or mixed with the others as a result of the transformation (6). For instance, we have shown in sect. 4 of ref. [8] how to obtain a magnetic spacetime from Minkowski spacetime upon performing a similarity transformation (6) on the latter.

For the case $\gamma = 0$, corresponding to pure KK theory, the whole set of exact solutions to the system (4 & 5)

have been systematically constructed, classified and interpreted as neutral or charged cosmic strings [8]. We found that two classes (7) among these solutions solve trivially (4 & 5) for $\gamma \neq 0$ since their Lanczos term vanishes identically; furthermore, they have been shown to be unique mathematical solutions up to linear coordinate transformations (6). A non-trivial solution with non-vanishing Lanczos term ($\gamma \neq 0$) has been derived by a perturbative approach, however, turned out to be exact and interpreted as a superconducting cosmic string (8). The purpose of this note is to derive the most general superconducting cosmic-string solution by directly integrating the non-linear field equations (4 & 5) under the same assumptions we made in our previous work and prove its unicity from a mathematical point of view, which has been left open so far.

For the case $\gamma \neq 0$, we mainly focused on a) the search for solutions depending on a scalar function $\omega(\rho)$ and a constant real matrix A of the form $\chi = \omega(\rho)A$. We have shown that the mathematically unique solution is [8]

$$\chi = (2/\rho)A, \quad \text{Tr} A = \text{Tr} A^2 = \text{Tr} A^3 = 1, \quad \text{Det} A = 0, \quad \text{with } r(A) = 1 \text{ or } r(A) = 2, \quad (7)$$

where $r(A)$ denotes the rank of A . The solution with rank 1 has been interpreted as a neutral cosmic string and that with rank 2 as a charged cosmic string. For both strings the Lanczos term vanishes identically so they are trivial solutions. Notice that because of (7) any similarity transformation on A , induced by a coordinate transformation of the form (6), results in a similarity transformation on χ and conversely. A result from matrix theory confirms that two matrices related by a similarity transformation have the same rank [27]. Since neutral and charged cosmic strings have different ranks so they cannot be related by a similarity transformation, consequently they do not belong to the same class of equivalence.

Then b) by a perturbative analysis the search for asymptotic solutions extending those derived in a). The perturbative analysis consisted in retaining the first few terms of the power series of χ in powers of $1/\rho$, that is, $\chi = (2/\rho)A + 2(\gamma/\rho^2)D + 2(\gamma/\rho^3)E$ where the first term $\chi_0 = (2/\rho)A$ is the unique exact solution found in a) and the 4×4 constant real matrices D & E had to be determined solving the non-linear system (4 & 5). We ended by finding the exact non-trivial solution

$$\chi = (2/\rho)A - (4\gamma/\rho^3)(A^3 - A^2), \quad (8)$$

with $\text{Tr} A = \text{Tr} A^2 = \text{Tr} A^3 = 1$, $\text{Det} A = 0$, $r(A) = 3$,

however, we failed to prove that it is mathematically the unique polynomial solution subject to the constraints (9). The perturbative approach has masked some features of the system that we will recover here. In fact, the non-linear system (4 & 5) can be directly integrated leading to the most general non-trivial solution (30). Solution (8) has been interpreted as an extended superconducting cos-

mic string surrounding a longitudinally boosted electrically charged naked cosmic string [8].

In both solutions (7) & (8) the invariants of A are subject to the constraints

$$\text{Tr}A = \text{Tr}A^2 = \text{Tr}A^3 = 1, \quad \text{Det}A = 0, \quad (9)$$

which reduce its characteristic equation to $A^4 = A^3$. A has then the eigenvalues 1, 0, 0 & 0 without necessarily being diagonalizable. Instead, the simplest form to which A can be brought by a similarity transformation is the Jordan normal form [27]

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{A} \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \\ 0 & 0 & 0 \end{pmatrix} \quad (10)$$

where $\epsilon_1, \epsilon_2 = 0$ or 1. Only the case $\epsilon_1 = 0$ & $\epsilon_2 = 0$ corresponding to $\text{r}(A) = 1$ is diagonalizable. The case $\text{r}(A) = 2$ corresponds either to $(\epsilon_1 = 0 \text{ \& } \epsilon_2 = 1)$ or to $(\epsilon_1 = 1 \text{ \& } \epsilon_2 = 0)$ and the case $\text{r}(A) = 3$ corresponds to $\epsilon_1 = \epsilon_2 = 1$. Now, with A constrained by (9), it is straightforward to check that $\text{r}(A) = 1$ or 2 implies $A^3 - A^2 = 0$, hence (7) is just a special case of (8) and the two solutions can be combined in one

$$\chi = (2/\rho)A - (4\gamma/\rho^3)(A^3 - A^2) \quad (11)$$

(without constraining $\text{r}(A)$), where the invariants of A are constrained by (9). The three cases with $\text{r}(A) = 1, 2$ or 3 do not belong to the same class of equivalence.

Exact solutions. – The solution shown in (11) is a special form of the more general case of $\chi(\rho)$ being polynomial in a constant real matrix A with scalar coefficients. Using the characteristic equation of A , one can eliminate any power of A higher than 3, hence the most general form of χ depending on a constant real matrix is a polynomial in A of third degree of the form

$$\chi(\rho) = \eta(\rho) + \omega(\rho)A + \beta(\rho)A^2 + \delta(\rho)A^3. \quad (12)$$

The constraints (9) on the invariants of A have ensured the existence of neutral, charged and superconducting cosmic strings so we will maintain them in our quest for new non-trivial cosmic strings and focus on the determination of the four real functions shown in (12). Since the system (4 & 5) is non-linear, it may admit different solutions than those shown in (11).

Case $\eta \equiv 0$. First we assume that $\eta(\rho) \equiv 0$ and reparameterize (12) by $\beta(\rho)$, $\delta(\rho)$ and $T(\rho) \equiv \text{Tr}\chi(\rho)$

$$\chi(\rho) = (T - \beta - \delta)(\rho)A + \beta(\rho)A^2 + \delta(\rho)A^3. \quad (13)$$

The set of conditions (9) implies the relation $A^n = A^3$ for $n \geq 3$ together with (13) reduce the system (4 & 5) to a scalar equation

$$P_5(\rho) \equiv 3[T_{,\rho} + (1/2)T^2] = 0, \quad (14)$$

along with a polynomial of degree 3 in A with scalar coefficients

$$P_4(\rho) + P_3(\rho)A + P_2(\rho)A^2 + P_1(\rho)A^3 = 0, \quad (15)$$

where

$$P_4 \equiv (2/3)P_5 \quad (16)$$

$$P_3 \equiv (1/3)P_5 - (1/2)T(\beta + \delta) - (\beta + \delta)_{,\rho} \quad (17)$$

$$P_1 + P_2 \equiv (1/2)T(\beta + \delta) + (\beta + \delta)_{,\rho} \quad (18)$$

$$\begin{aligned} P_1 \equiv & (\gamma/4)T^4 - (\gamma/2)(\beta + \delta)T^3 \\ & + [(3\gamma/2)T_{,\rho} - \gamma(\beta + \delta)_{,\rho} + (\gamma/4)(\beta + \delta)^2]T^2 \\ & + [(1/2)\delta - 2\gamma(\beta + \delta)T_{,\rho} + \gamma(\beta + \delta)(\beta + \delta)_{,\rho}]T \\ & + (\gamma/2)(\beta + \delta)^2T_{,\rho} + \delta_{,\rho}. \end{aligned} \quad (19)$$

Eq. (14) being satisfied, the polynomial (15) reduces taking into account the relations (16 to 18) to

$$P_3(\rho)A + P_2(\rho)A^2 + P_1(\rho)A^3 = 0, \quad (20)$$

$$P_1 + P_2 = -P_3 = (1/2)T(\beta + \delta) + (\beta + \delta)_{,\rho}, \quad (21)$$

and reduces further upon multiplying by A and using (21) along with $A^n = A^3$ for $n \geq 3$ to

$$P_3(\rho)(A^2 - A^3) = 0. \quad (22)$$

First, consider the case $P_3 \neq 0$ & $A^3 = A^2$ which leads upon substituting into (20) to $A^2 = A$. Now, the relations $A^3 = A^2 = A$, leading to $\text{r}(A) = 1$ or 2, reduce (13) to $\chi = TA$ of the form (7) already discussed.

We then expect new solutions from the other alternative

$$P_3 = 0 = -(1/2)T(\beta + \delta) - (\beta + \delta)_{,\rho} \quad \& \quad A^3 \neq A^2, \quad (23)$$

with necessarily $\text{r}(A) = 3$ ($\text{Det}A = 0$), leading taking into account (21) to

$$P_2 = -P_1. \quad (24)$$

Now, equations (23 & 24) together reduce (20) to $P_1 = 0 = -P_2$. Hence (14) along with the alternative (23) lead to the vanishing of the coefficients and the independent term of the polynomial (15). We will now proceed to the resolution of the equations $P_5 = 0$, $P_3 = 0$ & $P_1 = 0$; the other equations, $P_4 = 0$ & $P_2 = 0$, will be satisfied consequentially. For that end we need to rewrite $P_1 = 0$, where P_1 given by (19), in a simplified form using (14) to eliminate $T_{,\rho}$ and (23) to eliminate $(\beta + \delta)_{,\rho}$. Hence, when $P_5 = 0$ & $P_3 = 0$ are satisfied we have

$$P_1 = \delta_{,\rho} + (1/2)T\delta - (\gamma/2)T^2(T - \beta - \delta)^2 = 0. \quad (25)$$

Equation (14) has two solutions $T = 0$ & $T = 2/\rho$. If $T = 0$, (23) implies $\beta + \delta = c_1$ and (25) implies $\delta = c_2$. Hence, $\chi = -c_1A + (c_1 - c_2)A^2 + c_2A^3$ is a constant matrix case, which is already discussed in [8].

Consider the other solution for T : $T = 2/\rho$ where we have omitted an insignificant additive constant in the denominator. With this value of T , equation (23) too has two solutions enumerated 1) & 2) below.

1) A trivial solution to (23) is $\beta + \delta = 0$ which reduces (25) to

$$\delta_{,\rho} + (1/2)T\delta - (\gamma/2)T^4 = 0 \quad \text{with} \quad T = 2/\rho. \quad (26)$$

Integrating this last equation, we obtain introducing a real constant of integration $2K$

$$\delta = -(4\gamma/\rho^3) + 2K/\rho \quad \& \quad \beta = -\delta, \quad (27)$$

and then substituting into (13) we obtain

$$\chi = (2/\rho)A - (4\gamma/\rho^3 - 2K/\rho)(A^3 - A^2). \quad (28)$$

One sees that the solution (11) is a special case of (28).

2) The other solution to (23) is proportional to T given, introducing a real constant of integration $L \neq 0$, by $\beta + \delta = 2L/\rho = LT$. Substituting into (25) and introducing $\Gamma = \gamma(1-L)^2$ we obtain the equation $\delta_{,\rho} + (1/2)T\delta - (\Gamma/2)T^4 = 0$ (with $T = 2/\rho$) which is similar to (26) and hence has the same solution

$$\delta = -(4\Gamma/\rho^3) + 2K/\rho \quad \& \quad \beta = 2L/\rho - \delta, \quad (29)$$

Finally, substituting $T = 2/\rho$ and the corresponding expressions for δ & β shown in the previous equation into (13) we obtain

$$\begin{aligned} \chi &= (2/\rho)[A + L(A^2 - A) + K(A^3 - A^2)] \\ &\quad - [4\gamma(1-L)^2/\rho^3](A^3 - A^2). \end{aligned} \quad (30)$$

Introducing the two relevant physical constants $I = L - K$ & $V = 1 - L$, the two solutions (28 & 30) are combined and rewritten as:

Corollary. Assume A a 4×4 constant real matrix of arbitrary rank satisfying the conditions (9).

- i) If $M = mA + nA^2 + lA^3$ with $m + n + l = 1$, then M satisfies (9) and $r(M) = r(A)$.
- ii) If $M = VA + IA^2 + (1 - I - V)A^3$, I & $V \in \mathbb{R}$, then

$$\begin{aligned} \chi &= (2/\rho)M - (4\gamma V^2/\rho^3)(A^3 - A^2) \\ &= (2/\rho)M - (4\gamma/\rho^3)(M^3 - M^2) \end{aligned} \quad (31)$$

solves the system¹ (4 & 5). Since the rank of M (or A) is unconstrained, the different solutions are again classified according to its values leading to three classes of equivalence: neutral, charged and superconducting cosmic strings for $r(M) = 1$, $r(M) = 2$ & $r(M) = 3$, respectively.

Now starting from the non-trivial solution (11): $\chi = (2/\rho)A - (4\gamma/\rho^3)(A^3 - A^2)$ with A satisfying (9) & $r(A) = 3$ and performing a coordinate transformation to the new coordinates x_N (6), the thus diffeomorphically obtained solution remains form-invariant: $\chi_N = (2/\rho)A_N - (4\gamma/\rho^3)(A_N^3 - A_N^2)$ where

¹The cases $L = 0$ ($V = 1$) & $L \neq 0$ ($V \neq 1$) correspond to (28) & (30), respectively.

$A_N = S^{-1}AS$, $\chi_N = S^{-1}\chi S$. Since this latter solution χ_N is of the form (31), one may try to solve for S such that $M \equiv A_N = S^{-1}AS$. Taking $M = VA + IA^2 + (1 - I - V)A^3$ and A of the form (10) with $\epsilon_1 = \epsilon_2 = 1$ one obtains $S^2_2 = S^3_3 = 1$, $S^4_4 = V$, $S^5_5 = V^2$, $S^4_5 = I$ and the other elements S^a_b vanish. Hence, with (9) satisfied and $\eta(\rho) \equiv 0$, the system (4 & 5) admits no further solutions of the form (12) except those obtained by coordinate transformations (6) from (11). Although the two solutions (11) & (31) are mathematically diffeomorphic, however, they are only locally equivalent. The similarity transformation S derived above has the components $S^5_5 \neq \delta^5_5 = 1$ & $S^4_5 \neq \delta^4_5 = 0$, this results in re-scaling the Killing vector ξ_5^A , which has closed orbits, and mixing it with ξ_4^A . Hence, the new solution (31) is not globally equivalent to (11) [8].

Case $\eta \neq 0$. Let us now assume that $\eta(\rho) \neq 0$. We will show that the system (4 & 5) does not admit any solution of the form (12) if the constraints (9) are satisfied. Since the latter lead to $A^n = A^3$ for $n \geq 3$ we have then $\text{Tr}A^m = 1$ for $m \geq 1$ and consequently $A^m \neq 0$ for $m \geq 1$. Now, substituting (9 & 12) into the system (4 & 5) and using $A^n = A^3$ for $n \geq 3$, the latter reduces to

$$\mathbb{P}_5(\rho) = 0; \quad (32)$$

$$\mathbb{P}_4(\rho) + \mathbb{P}_3(\rho)A + \mathbb{P}_2(\rho)A^2 + \mathbb{P}_1(\rho)A^3 = 0, \quad (33)$$

where \mathbb{P}_1 to \mathbb{P}_5 are expressed in terms of the unknown functions η , ω , β & δ . Multiplying (33) by A^3 and using once more $A^n = A^3$ for $n \geq 3$ along with $A^m \neq 0$ for $m \geq 1$, we obtain

$$\mathbb{P}_4 + (\mathbb{P}_3 + \mathbb{P}_2 + \mathbb{P}_1) = 0. \quad (34)$$

The trace of (33) leads to

$$4\mathbb{P}_4 + (\mathbb{P}_3 + \mathbb{P}_2 + \mathbb{P}_1) = 0. \quad (35)$$

The homogeneous system (34 & 35) in the two variables \mathbb{P}_4 & $(\mathbb{P}_3 + \mathbb{P}_2 + \mathbb{P}_1)$ has only trivial solutions

$$\mathbb{P}_4 = 0 \quad \& \quad (\mathbb{P}_3 + \mathbb{P}_2 + \mathbb{P}_1) = 0. \quad (36)$$

Now, it is straightforward to show that the system of three equations (32 & 36) does not admit any solution if $\eta(\rho) \neq 0$. In (12), re-parameterizing χ by η , $\sigma = \omega + \beta + \delta$, β & δ , the system of three equations (32 & 36) writes

$$\begin{aligned} \mathbb{P}_5 &\equiv [12\eta + 3\sigma + (3/2)\gamma\sigma\eta^2 + 2\gamma\eta^3]_{,\rho} \\ &\quad + 12\eta^2 + 6\eta\sigma + (3/2)\sigma^2 \end{aligned} \quad (37)$$

$$\begin{aligned} &+ 3\gamma\eta^2[\eta^2 + \eta\sigma + (1/4)\sigma^2] = 0; \\ \mathbb{P}_4 &\equiv [9\eta + 2\sigma + (5/2)\gamma\eta^3]_{,\rho} + \gamma[\eta^2\sigma_{,\rho} + 7\sigma(\eta^2)_{,\rho}] \\ &\quad + 12\eta^2 + (11/2)\eta\sigma + \sigma^2 \end{aligned} \quad (38)$$

$$\begin{aligned} &+ \gamma\eta^2[12\eta^2 + (35/4)\eta\sigma + (5/4)\sigma^2] = 0; \\ \mathbb{P}_3 + \mathbb{P}_2 + \mathbb{P}_1 &\equiv \sigma_{,\rho} + (1/2)\sigma^2 + 2\eta\sigma \\ &\quad + \gamma[3\eta(\sigma^2)_{,\rho} + (7/2)\eta^2\sigma_{,\rho} + (1/2)\sigma(\eta^2)_{,\rho}] \\ &\quad + 3\gamma\eta\sigma[13\eta^2 + (43/4)\eta\sigma + (9/4)\sigma^2] = 0. \end{aligned} \quad (39)$$

The three first order differential equations (37 to 39), which do not depend on β and δ , have to be solved for η & σ . When $\eta(\rho) \neq 0$, this system will not admit any solution unless one of the three equations is a combination of the two others. A way to check that is to solve for the derivatives $\eta_{,\rho}$ & $\sigma_{,\rho}$ using (37) & (39) then substitute them into (38) whose expression has to vanish identically. We have checked it directly and found out that the expression of \mathbb{P}_4 (38) reduces, but does not vanish identically, to

$$\begin{aligned} 2\gamma^2\mathbb{P}_4 = & \{3\gamma^2\eta(\gamma\eta^2 + 1)(2\eta + \sigma)^4 \\ & + \gamma(5\gamma^2\eta^4 + 5\gamma\eta^2 + 2)(2\eta + \sigma)^3 \\ & + 4\gamma\eta(5\gamma\eta^2 - \gamma^2\eta^4 + 6)(2\eta + \sigma)^2 \\ & + (3\gamma^3\eta^6 + 6\gamma^2\eta^4 + 20\gamma\eta^2 + 8)(2\eta + \sigma) \\ & - \gamma\eta^3(\gamma^2\eta^4 + 4\gamma\eta^2 + 4)\}/[\eta\sigma(2\eta + \sigma)]. \end{aligned} \quad (40)$$

The superconducting cosmic string. – The 5-metric (3) is found upon integrating $\chi \equiv \lambda^{-1}\lambda_{,\rho}$ using (30) or (31)

$$\begin{aligned} \lambda = & C\{1 - M^3 + \rho^2 M^3 - 2\ln\rho(M^3 - M) \\ & - 2[(\ln\rho)^2 - \gamma/\rho^2](M^3 - M^2)\} \\ = & C\{1 - A^3 + \rho^2 A^3 + 2\ln\rho[VA + IA^2 - (I + V)A^3] \\ & - 2V^2[(\ln\rho)^2 - \gamma/\rho^2](A^3 - A^2)\}. \end{aligned} \quad (41)$$

where C is a constant real matrix of signature $(- - + -)$. Since λ is symmetrical, C satisfies the relations $C = C^T$ & $CA = (CA)^T$ (T denotes transpose). Choosing A of the form (10) with $\epsilon_1 = \epsilon_2 = 1$ ($r(A) = 3$) & C of the form

$$C = \begin{pmatrix} -\alpha^2 & 0 \\ 0 & \mathbb{C} \end{pmatrix}, \quad \mathbb{C} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & a \\ -1 & a & b \end{pmatrix} \quad (42)$$

the 5-metric (3) writes

$$\begin{aligned} ds^2 = & -d\rho^2 - \alpha^2\rho^2 d\varphi^2 - 2dz dx^5 \\ & - dt^2 - 4VQ(\rho) dt dx^5 \\ & - 2[V^2Q(\rho)^2 + IQ(\rho) - \gamma V^2/\rho^2 - p](dx^5)^2, \end{aligned} \quad (43)$$

where $p = (a^2 + 2b)/4 - (aI/2V)$, $Q(\rho) = \ln(\rho/\rho_0)$ and ρ_0 is such that $a = 2V \ln \rho_0$.

The metric (43) is subject to the following interpretation. As we did earlier [8], reinterpreting the Lanczos term $-(\gamma/8\pi G)L^A_B$ as a 5-dimensional effective energy-momentum tensor source, equation (1) writes $R^A_B - (1/2)R\delta^A_B = 8\pi GT^A_B$ where we have inserted the gravitational constant. We evaluate the components of R^A_B as

$$R^\mu_\nu = (V\mathbb{A} + I\mathbb{A}^2)\Delta(\ln\rho) - \gamma V^2\mathbb{A}^2\Delta(1/\rho^2) \quad (44)$$

$$R^1_1 = R^2_2 = [(\alpha - 1)/\alpha]\Delta(\ln\rho) \quad (45)$$

where Δ is the covariant Laplacian with respect to the metric $d\rho^2 + \alpha^2\rho^2 d\varphi^2$ (μ & ν assume the values from 3 to 5). Using the relations $\Delta(1/\rho^2) = 4/\rho^4$

& $\Delta(\ln\rho) = 2\pi\alpha\delta^2(\mathbf{x})$, where $\mathbf{x} = \sqrt{\rho}(\cos\varphi, \sin\varphi)$ and $\delta^2(\mathbf{x})$ is the 2-dimensional covariant distribution, we obtain the non-vanishing components of the 5-dimensional effective energy-momentum tensor source by (since $\text{Tr}\mathbb{A} = \text{Tr}\mathbb{A}^2 = 0$ then $R = R^A_A = 2R^1_1$):

$$T^3_3 = T^4_4 = T^5_5 = \frac{(1 - \alpha)}{4G}\delta^2(\mathbf{x}) \quad (46)$$

$$T^3_4 = T^4_5 = \frac{\alpha V}{4G}\delta^2(\mathbf{x}) \quad (47)$$

$$T^3_5 = \frac{\alpha I}{4G}\delta^2(\mathbf{x}) - \frac{\gamma V^2}{2\pi G\rho^4}. \quad (48)$$

Hence, the tension (T^3_3), mass (T^4_4), scalar charge (T^5_5), electric charge (T^4_5) and momentum (T^3_4) densities of the string are purely distributional. As is the case for local strings, there is no tension in the radial direction ($T^1_1 = 0$) nor a pressure in the φ -direction ($T^2_2 = 0$). The longitudinal current density T^3_5 has a distributional contribution as well as a continuous contribution proportional to V^2 , which diverges as ρ approaches zero. For $V \neq 0$, the 5-metric (43) is then interpreted as an extended superconducting cosmic string (continuous contribution of $T^3_5 \rightarrow \infty$ as $\rho \rightarrow 0$) surrounding a naked electrically charged cosmic string core in longitudinal translation with a speed proportional to V and carrying an electric current proportional to I .

Conclusion. – Under cylindrical symmetry assumption, the field equations of KK theory extended by GB term reduce to the system (4 & 5) of non-linear matrix and scalar differential equations governing the behavior of a 4×4 real matrix $\chi(\rho)$ whose elements are functions of the 5-metric and its derivative. The system (4 & 5) may admit a variety of different solutions whose quest is still an open question, in this letter we have shown that if $\chi(\rho)$ is expressed as a polynomial in a constant real matrix A and if this is constrained by $\text{Tr}A = \text{Tr}A^2 = \text{Tr}A^3 = 1$ & $\text{Det}A = 0$, then the system (4 & 5) admits an exact unique solution up to linear coordinate transformations mixing its Killing vectors. Since the rank of A is unconstrained, the different solutions are classified according to its values in such a way that solutions with the same rank of A belong to the same class of equivalence. Neutral and charged cosmic strings correspond to $r(A) = 1$ & $r(A) = 2$, respectively, while the generic case $r(A) = 3$ corresponds to superconducting cosmic strings. 4-stationary solutions belonging to same class of equivalence are not all globally equivalent due to the existence of two Killing vectors with closed orbits. For $r(A) = 3$, we have shown that the most general superconducting cosmic string is in arbitrary translational motion parallel to its axis, carries an arbitrary electric current and is surrounded by a continuous longitudinal current density.

The question whether the non-linear system (4 & 5) admits other polynomial solutions in a constant matrix A , however, constrained by other conditions than those shown above is still an open topic. On the other hand,

there is no reason to restrict oneself to solutions depending on a constant matrix; are there solutions of different shapes, for instance solutions depending on two constant matrices? In a project that we are realizing, these questions are only partly answered: we managed to prove that any solution $\chi(\rho)$ to the system (4 & 5) is necessarily a polynomial in a constant real matrix with scalar coefficients depending on ρ [28]. This restricts the search for new solutions to polynomials.

REFERENCES

- [1] LOVELOCK D., *J. Math. Phys.*, **12** (1971) 498.
- [2] ZWIEBACH B., *Phys. Lett.*, **156B** (1985) 315;
BOULWARE D.G. and DESER S., *Phys. Rev. Lett.*, **55** (1985) 2656.
- [3] ZUMINO B., *Phys. Reports*, **137** (1986) 109.
- [4] MADORE J., *Phys. Lett.*, **110A** (1985) 289.
- [5] MÜLLER-HOISSEN F., *Phys. Lett.*, **163B** (1985) 106.
- [6] DERUELLE N. and MADORE J., *Mod. Phys. Lett.*, **A1** (1986) 237.
- [7] WHEELER J.T., *Nucl. Phys.*, **B268** (1986) 737.
- [8] AZREG-AÏNOU M. and CLÉMENT G., *Class. Quantum Grav.*, **13** (1996) 2635.
- [9] AYÓN-BEATO E. and GARCÍA A., *Phys. Rev. Lett.*, **80** (1998) 5056.
- [10] AYÓN-BEATO E. and GARCÍA A., *Gen. Relativ. Gravit.*, **37** (2005) 635.
- [11] BURINSKII A. and HILDEBRANDT S.R., *Phys. Rev.*, **D 65** (2002) 104017.
- [12] BUCHDAHL H.A., *J. Phys. A: Math. Gen.*, **12** (1979) 10370.
- [13] WILTSHIRE D.L., *Phys. Lett.*, **169B** (1986) 36;
WILTSHIRE D.L., *Phys. Rev.*, **D 38** (1988) 2445.
- [14] THIBEAULT M., SIMEONE C. and EIROA E.F., *Gen. Relativ. Gravit.*, **38** (2006) 1593.
- [15] CAI R.-G., *Phys. Lett.*, **582B** (2004) 237;
IZAURIETA F., RODRIGUEZ E. and SALGADO P., *Phys. Lett.*, **586B** (2004) 397;
AIELLO M., FERRARO R. and GIRIBET G., *Phys. Rev.*, **D 70** (2004) 104014.
- [16] BHAWAL B. and KAR S., *Phys. Rev.*, **D 46** (1992) 2464.
- [17] DOTI G., OLIVA J. and TRONCOSO R., *Phys. Rev.*, **D 75** (2007) 024002.
- [18] FARAKOS K. and PASIPOULARIDES P., *Phys. Rev.*, **D 75** (2007) 024018.
- [19] MAEDA H. and DADHICH N., *Phys. Rev.*, **D 75** (2007) 044007.
- [20] GRAVANIS E. and WILLISON S., *Phys. Rev.*, **D 75** (2007) 084025.
- [21] MELIS M. and MIGNEMI S., *Phys. Rev.*, **D 75** (2007) 024042.
- [22] COGNOLA G., ELIZALDE E., NOJIRI S., ODINTSOV S.D. and ZERBINI S., *Phys. Rev.*, **D 75** (2007) 086002.
- [23] KONYA K., *Phys. Rev.*, **D 75** (2007) 104003.
- [24] BARROW J.D. and MIDDLETON J., *Phys. Rev.*, **D 75** (2007) 123515.
- [25] BEREJ W., MATYJASEK J., TRYNIECKI D. and WORONOWICZ M., *Gen. Relativ. Gravit.*, **38** (2006) 885.
- [26] ORTÍN T., *Gravity and Strings* (Cambridge University Press, West Nyack, NY) 2004;
- SAZHIN M. *et al.*, *Mon. Not. R. Astron. Soc.*, **343** (2003) 353.
- [27] BARNETT S., *Matrices: Methods and Applications* (Clarendon Press, Oxford) 1990;
LANCASTER P. and TISMENETSKY M., *The Theory of Matrices: with Applications* (Academic Press) 1985.
- [28] AZREG-AÏNOU M., “Solving a class of non-linear matrix differential equations with application to General Relativity,” in preparation.